PENANG SANGAM HIGH SCHOOL
P.O.BOX 44, RAKIRAKI
LESSON NOTES – WEEK 16-18

School: Penang Sangam High School **Year/Level:** 13 **Subject:** Mathematics

Strand	4 **TRIGONOMETRY**
Sub Strand	4.2.1 Trigonometric Graphs
Content Learning Outcome	Students should be able to; - Draw trigonometric graphs

Trigonometric Graphs

The general form is defined as

$$y = A\,Sin\,(Bx \pm C) \pm k \qquad or \qquad y = A\,Cos\,(Bx \pm C) \pm k$$

$$y = A \quad \begin{matrix}Sin\\Cos\end{matrix} \quad (B\,x \pm C) \quad \pm \quad k$$

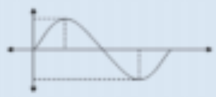
Amplitude: The height / distance

$+A$ means the graph is oriented as usual

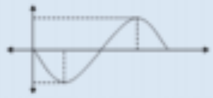
$-A$ means that the graph is inverted

Shape:

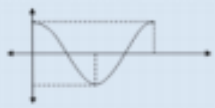
+ A Sin



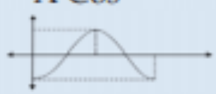
− A Sin



+ A Cos



− A Cos



Period:

B helps determine the *period* of the graph (the length of the interval needed for the graph of the function to start repeating itself).

$$period = \frac{360}{B} \quad or \quad \frac{2\pi}{B}$$

C shifts the y-axis or the graph by $\frac{C}{B}$ units.

$+\frac{C}{B}$ shifts the y-axis to the right or the graph moves by $\frac{C}{B}$ units to the left.

$-\frac{C}{B}$ shifts the y-axis to the left or the graph moves by $\frac{C}{B}$ units to the right

$\boldsymbol{k}$ shifts the x-axis or the graph by $\boldsymbol{k}$ units.

$+\boldsymbol{k}$ shifts the x-axis down by $\boldsymbol{k}$ units or the graph moves up by $\boldsymbol{k}$ units

$-\boldsymbol{k}$ shifts the x-axis up by $\boldsymbol{k}$ units or the graph moves down by $\boldsymbol{k}$ units

Example 1:

A trigonometric function is defined as $f(x) = 3\sin\left(x + \frac{\pi}{4}\right)$

i) Write the period of the function $f(x)$

$$y = A\sin(Bx \pm C) = \pm k$$

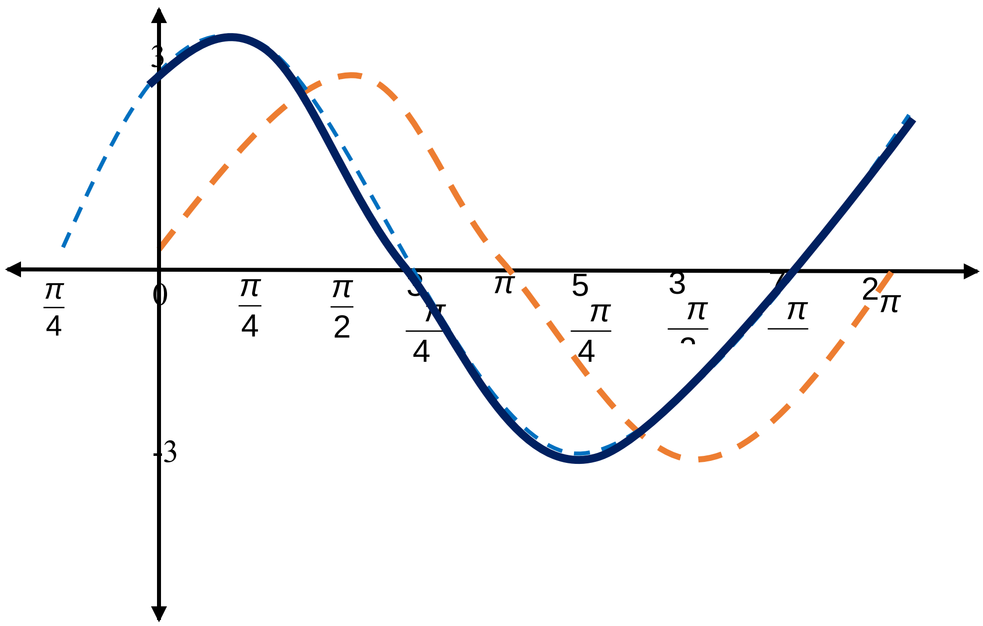
$$f(x) = 3\sin\left(1x + \frac{\pi}{4}\right) \qquad \text{Period} = \frac{2\pi}{B} = \frac{2\pi}{1} = 2\pi$$

ii) What is the Amplitude of the function?

Amplitude: A = 3

iii) Sketch $f(x) = 3\sin(x + \frac{}{4})$ *for* $0 \le x \le 2$

Shift the graph $\frac{\pi}{4}$ units to the left or shift the y- axis $\frac{\pi}{4}$ units to the right



iv) Write down the coordinates of the maximum point of *f(x)* **for the region** $0 \le x \le 2$

$$\left(\frac{\pi}{4}, 3\right)$$

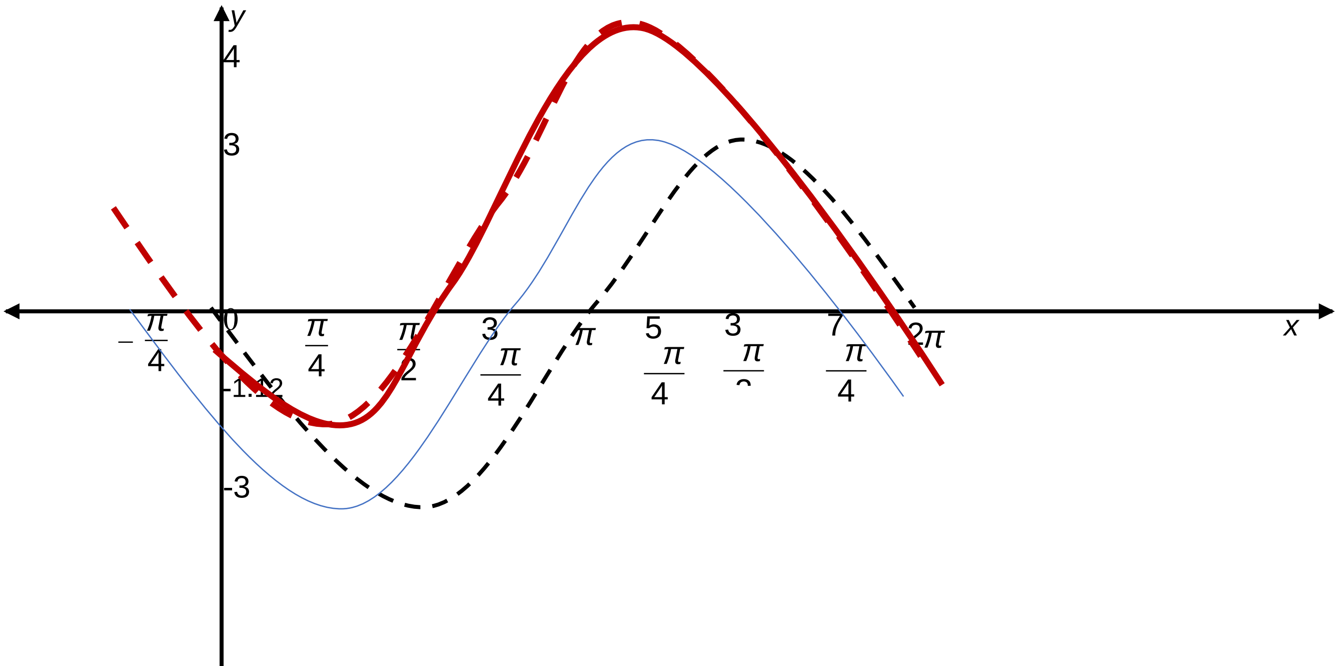
Example 2:

Sketch the graph of $f(x) = -3\sin(x + \frac{}{4}) + 1$

Amplitude: 3 Period : $\frac{2\pi}{1} = 2\pi$

y int (let $x = 0$) $y = -3\sin(x + \frac{\pi}{4}) + 1$

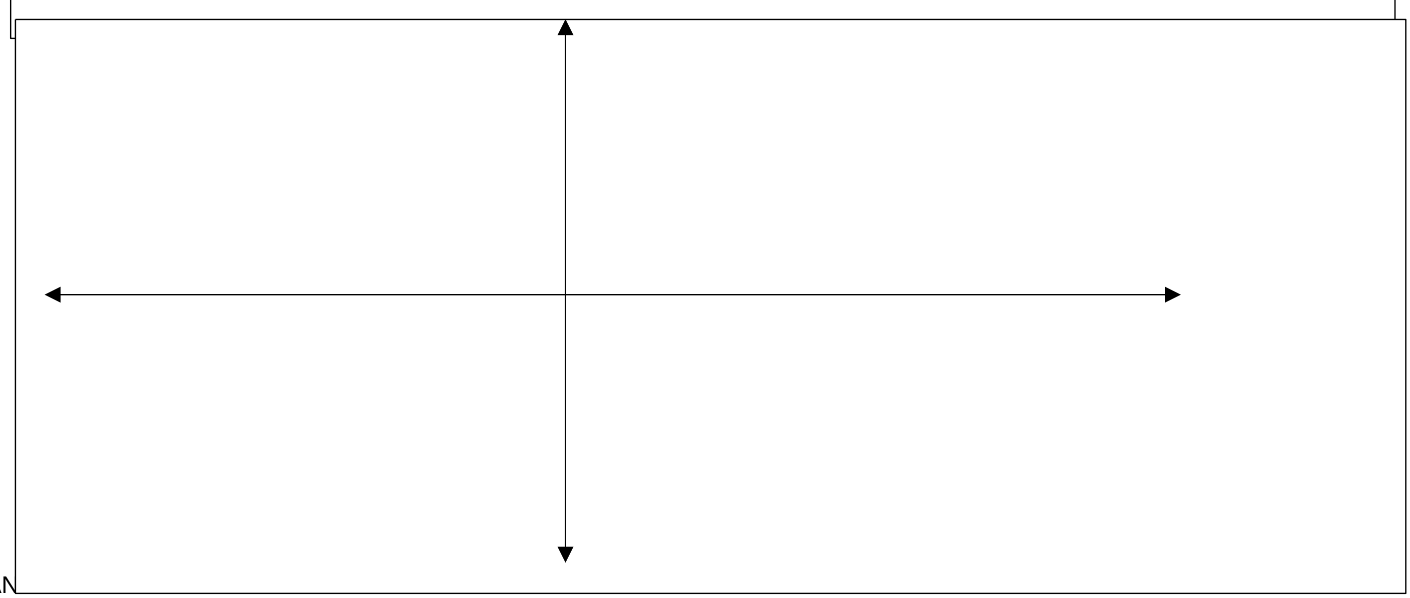
$y = -3\sin(0 + \frac{\pi}{4}) + 1$



Exercise:

1. **Sketch the following graph.**

$$y = 2\cos(x + 90^\circ) - 2$$



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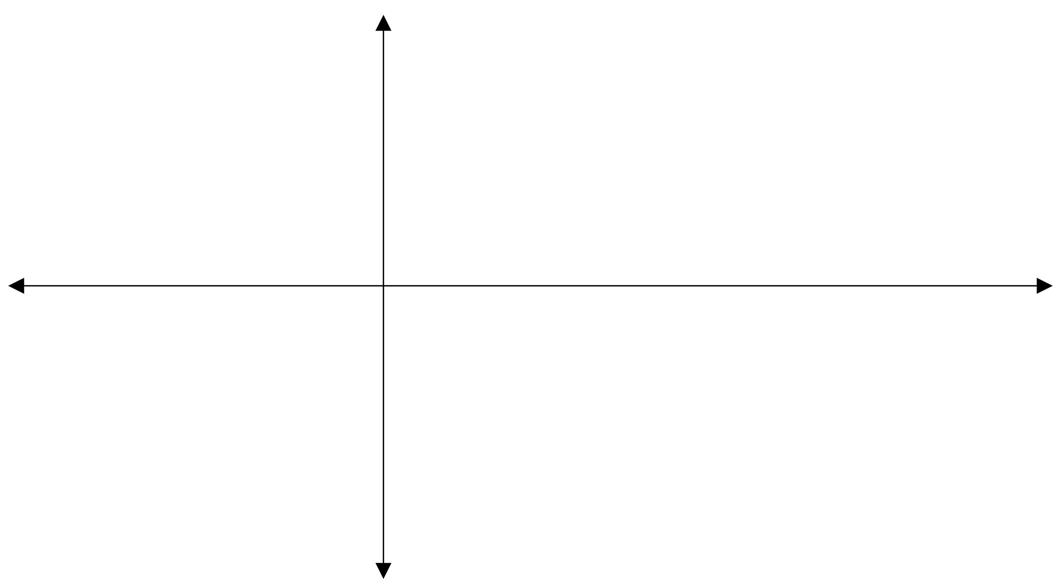
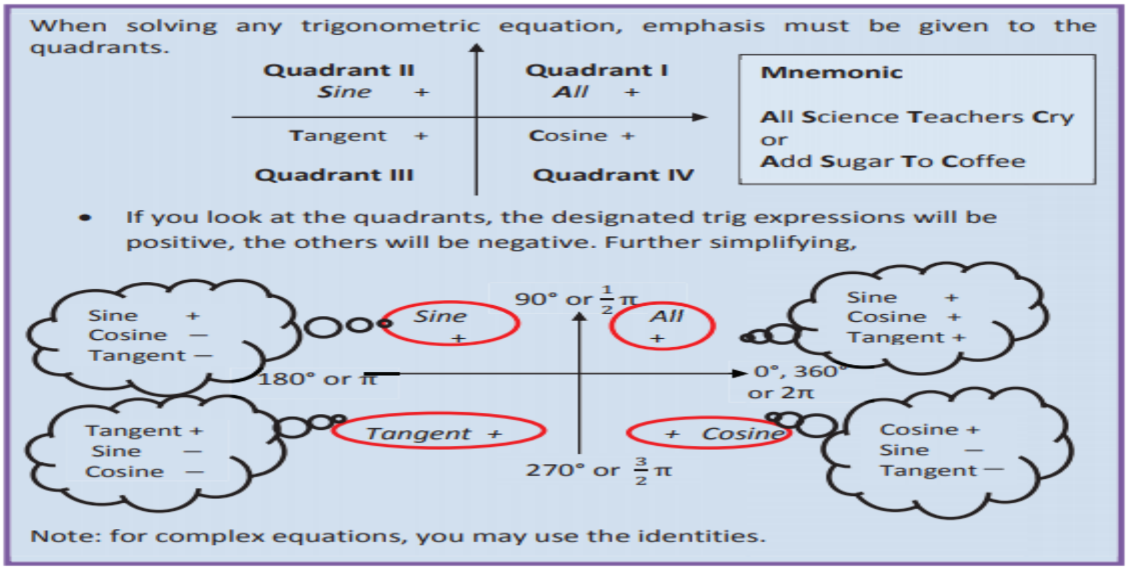
ii)Sketch the graph of $y = -2\cos(x - 90^{\circ}) + 1$



Strand	4 **TRIGONOMETRY**
Sub Strand	4.3.1 Trigonometric Equations
Content Learning Outcome	Students should be able to; - solve trigonometric equations

Solving Trigonometric Equations



Example1:

Solve for the following trigonometric equations.

1. $\cos 2 \ + \cos \ + 1 = 0$ for $0 \le \ \le 2$

$\cos 2\theta = 2\cos^2\theta - 1$

$2\cos^2\theta - 1 + \cos\theta + 1 = 0$

$2\cos^2\theta \cancel{- 1} + \cos\theta \cancel{+ 1} = 0$

$2\cos^2\theta + \cos\theta = 0$

$\cos\theta\,(2\cos\theta + 1) = 0$

$2\cos\theta + \cancel{1 - 1} = 0 - 1$

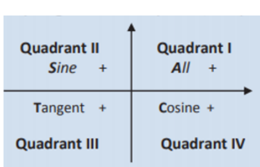
$\cos\theta = 0$

$2\cos\theta + 1 = 0$

$2\cos\theta = -1$

$= \{90^\circ, 270^\circ\}$

$\cos\theta = \dfrac{-1}{2}$



$Q3 = 180^\circ + \theta$

$= 180^\circ + 60^\circ$

Since cos is negative it falls in quadrant II and III

$Q2 = 180^\circ - \theta$

2. $\sin 2 \ = \sin \ $ for $0^\circ \le \ \le 360^\circ$

$\sin 2\theta = \sin\theta$

$\sin 2\theta - \sin\theta = 0$

$\boxed{\sin 2\theta = 2\sin\theta\cos\theta}$

$2\sin\theta\cos\theta - \sin\theta = 0$

$\sin\theta(2\cos\theta - 1) = 0$

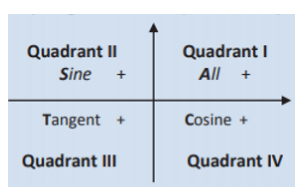
$2\cos\theta - 1 + 1 = 0 + 1$

$\sin\theta = 0$

$2\cos\theta - 1 = 0$

$\dfrac{2\cos\theta}{2} = \dfrac{1}{2}$

$= \{0^\circ, 180^\circ, 360^\circ\}$



Since cos is positive θ falls

$Q1 = \theta$

$\theta = \{60^\circ\}$

$Q4 = 360^\circ - \theta$

$= 360^\circ - 60^\circ$

$= 360^\circ$

$Q4 = 360^\circ - 360^\circ$

2. $2\,cos^2 = \sin 2$

Strand	4	**TRIGONOMETRY**
Sub Strand	**4.4.1 Applications of addition formulae**	
Content Learning Outcome	Students should be able to; - Use addition formulae to write a sum of sine and cosine as either a sine or cosine function - Find maximum and minimum points on a given interval	

Often trig expressions involve the sum of sine and cosine terms. It is more convenient to write such expressions using one single term by applying the addition formula:

(i) $a\,Cos\,\theta \pm b\,Sin\theta = r\,Cos\,(\theta \pm \alpha)$

(ii) $a\,Cos\,\theta \pm b\,Sin\theta = r\,Sin\,(\theta \pm \alpha)$

where α is an angle to be found and r is the modulus i.e. $r = \sqrt{a^2 + b^2}$ and a and b are ***coefficients*** of $Cos\ \theta$ and $Sin\ \theta$ respectively.

Example1:

Write $y = 3\sin \quad - 4\cos$ **in the form of** $y = r\sin(\quad + \quad)$

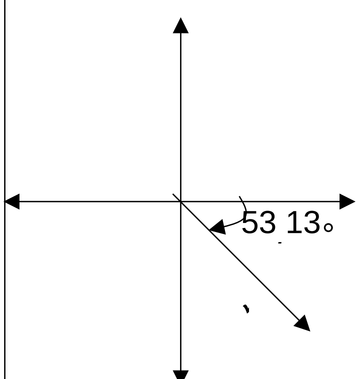
$3\sin\theta - 4\cos\theta = r\sin(\theta + \alpha)$

$3\sin\theta - 4\cos\theta = 5\sin(\theta + \alpha)$

$r = 5$

$\boxed{a\,Cos\,\theta \pm b\,Sin\theta = r\,Sin\,(\theta \pm \alpha)}$

- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
- $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$



$3\sin\theta - 4\cos\theta = 5(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$

$-4\cancel{\cos\theta} = 5\cancel{\cos\theta}\sin\alpha$

$3\sin\theta - 4\cos\theta = 5\sin\theta\cos\alpha + 5\cos\theta\sin\alpha$

$-4 = 5\sin\alpha$

$3\sin\theta = 5\sin\theta\cos\alpha$

$$\frac{-4}{5} = \sin\alpha$$

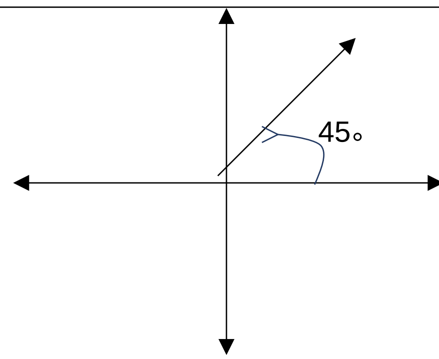
$3 = 5\cos\alpha$

$$\frac{3}{5} = cos\alpha$$

A function is given as

a) Express $f(x)$ in the form $f(x) = r\cos(\ - \ )$, where is an acute angle.

$$\boxed{\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B}$$



$\alpha = 45°$

$\alpha = 45°$

$f(x) = 2\cos(\ - 45°)$

ii) Sketch the graph of for the interval $0 \le x \le 2$

To sketch the graph;

- Plot the amplitude on the y -axis
- Mark 5 points (minus will indicate plotting starts from positive x-axis while plus will be negative x - axis)
- Add 90° to the angle for each interval
- Find y – intercept
- Draw solid line for the graph.

y-intercept ($\theta = 0$)

$y = 2\cos(0 - 45°)$

max $T.P$

$y = 2\cos(\theta - 45°)$

$2 = 2\cos(\theta - 45°)$

$1 = \cos(\theta - 45°)$

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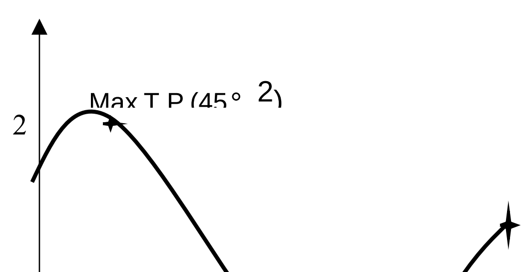

360∘

c) Give the coordinates of the minimum and the maximum point on

for the interval $0 \leq x \leq 2$

$f(x) = 2\cos(\; - 45^\circ)$

$2 = 2\cos(\; - 45^\circ)$

$\frac{2}{2} = \frac{2\cos(\; - 45^\circ)}{2}$

$1 = \cos(\; - 45^\circ)$

$(\; - 45^\circ) = \cos^{-1} 1$

$(\; - 45^\circ) = 0$

$f(x) = 2\cos(\; - 45^\circ)$

$-2 = 2\cos(\; - 45^\circ)$

$\frac{-2}{2} = \frac{2\cos(\; - 45^\circ)}{2}$

$-1 = \cos(\; - 45^\circ)$

$(\; - 45^\circ) = \cos^{-1} -1$

$45 \;) = 180$

d) Solve the equationfor $0^\circ \leq x \leq 360^\circ$

$\theta = 45^\circ$

$2\cos(\; - 45^\circ) = 1 \quad (45^\circ, 2)$

$\cos(\; - 45^\circ) = \frac{1}{2}$

Cos is positive: falls in Q1 and Q4

$(\theta - 45^\circ) = \cos^{-1}\frac{1}{\;}$

$Q4 = 360^\circ - \theta$

$360^\circ - 60^\circ = 330^\circ$

$(\theta - 45^\circ) = 330^\circ$

$\theta = 330^\circ + 45^\circ$

Since $\theta = 375^\circ$ is outside the interval, the answer will be

$\{105^\circ\}$

Exercise:

1. A function is given as $f(x) = 7\cos x - 6\sin x.$

a) **Express the function** $f(x)$ **in the form** $f(x) = R\cos(x + \quad)$ **where is an acute angle**.

ii) Sketch the graph of $f(x) = 7\cos x - 6\sin x$ **for** $0^\circ \leq x \leq 360^\circ$

iii. Solve the equation $7\cos x - 6\sin x = 5$ **for** $0^\circ \leq x \leq 360^\circ$

2. A function is defined by $f(x) = 4\cos x + 7\sin x$**.**

i) Express the function $f(x)$ **in the form** $f(x) = R\sin(x + \quad)$ **where is an acute angle.**

iii) **Solve the equation** $4\cos x + 7\sin x = -5 \quad 0^\circ \le x \le 360^\circ$

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Strand	5 **LIMITS CONTINUITY AND DIFFERENTIABILITY**
Sub Strand	5.5.1 Computing Limits
Content Learning Outcome	Students should be able to; - Calculate limits

Computing Limits

- $\lim_{x \to 0} f(x)$ is read as limit of $f(x)$ as x approaches 0
- The limit of a function $f(x)$ as "$x = a$" exists if the value of a the function approaches the same value as we get closer to a from both sides.

Direct Substitution

Result when x is substituted	Conclusion	Example

Sensible Answer	Answer becomes limit	$\lim_{x \to 2} x + 3 = 2 + 3$ $= 5$
$\dfrac{0}{\text{Number}}$	Answer is 0	$\lim_{x \to -5} \dfrac{x+5}{x+2} = \dfrac{-5+5}{-5+2} = \dfrac{0}{-3} = 0$
$\dfrac{\text{Number}}{0}$	- No limit - There is no limit - Limit does not exists	$\lim_{x \to 3} \dfrac{x+3}{x-3} = \dfrac{3+3}{3-3} = \dfrac{6}{0}$ No limit
$\dfrac{0}{0}$ (Algebraic Manipulation)	- Stop - Factorise - Cancel - Try again	$\lim_{x \to 6} \dfrac{x-6}{x^2-36} = \dfrac{6-6}{6^2-36} = \dfrac{0}{0}$ Then factorise $\lim_{x \to 6} \dfrac{x-6}{(x-6)(x+6)} =$ $\lim_{x \to 6} \dfrac{1}{x+6} = \dfrac{1}{6+6} = \dfrac{1}{12}$

L' Hôpitals Rule

- Used for situation whereby the answer is $\frac{0}{0}$ upon direct substitution

- If upon substitution $\frac{f(x)}{g(x)}$ gives $\frac{0}{0}$

- Differentiate f and g separately.
- Find the limit of $\frac{f'(x)}{g'(x)}$, by substituting the value.

Example:

$$\lim_{x \to 6} \frac{x - 6}{x^2 - 36} = \frac{6 - 6}{6^2 - 36} = \frac{0}{0}$$

Then differentiate the numerator and the denominator

$$\lim_{x \to 6} \frac{1}{2x} = \frac{1}{2(6)} = \frac{1}{12}$$

Example 1: **Find** $\lim_{x \to 0} \sin x$ **Example 2: Find** $\lim_{x \to -2} \frac{x + 2}{x - 3}$ **Example 3: Evaluate** $\lim_{x \to 3} \frac{x^2 + 2x + 1}{x - 3}$

Directly substitute 0, $\lim_{x \to 0} \sin x$ Directly substitute -2, $\lim_{x \to -2} \frac{x + 2}{x - 3}$ Directly substitute 2, = $\frac{3^2 + 2(3) + 1}{3 - 3}$

$= \sin 0$

$= 0$

$= \dfrac{-2+2}{-2-3}$

$= \dfrac{0}{-5} = 0$

<u>Exercise</u>

a. $\lim_{x\to 3} \dfrac{2x^2-5x-3}{x-3}$

b. $\lim_{x\to 10} x^2-10$

c. $\lim_{x\to 0} \cos x$

d) $\lim_{x\to 1} (3x^2+2x-6)$

e) $\lim_{x\to 2} \dfrac{3x^2+1}{x-3}$

f) $\lim_{x\to -1} \dfrac{2x-6}{3x+2}$

More Examples

Example 1:

Evaluate $\lim_{x \to 1} \frac{1-x^2}{1-x}$

Method 1: L'Hopitals Rule

$= \lim_{x \to 1} \frac{1-x^2}{1-x}$

$= \lim_{x \to 1} \frac{-2x}{-1}$

$= \frac{-2 \times 1}{-1}$

$= 2$

Method 2: Algebraic Manipulation

First we have to factorise, cancel and then substitute the value

$= \lim_{x \to 1} \frac{1-x^2}{1-x}$

Factorise $1-x^2$, using difference of squares gives $(1-x)(1+x)$

$= \lim_{x \to 1} \frac{(1-x)(1+x)}{1-x}$

$= \lim_{x \to 1} 1+x$

$= 1+1 = 2$

Method 3: Using the table

In this method we are going to take values that are closer to 1 from both the sides. Then we are going to put it in place of x and check the answer to what value is it approaching form both the sides.

x	0.97	0.98	0.99	1	1.01	1.02	1.03
$\frac{1-x^2}{1-x}$	1.97	1.98	1.99	undefined	2.01	2.02	2.03

It turns out that as x approaches 1, f(x) approaches 2

Thus, $\lim_{x\to 1} \frac{1-x^2}{1-x} = 2$

Example 5: Evaluate

Ans: *Directly substitute the value of 9,*

Evaluate =

Since it is $\frac{0}{0}$, *therefore we need to move to the next step. Looking at the previous example, we can use either of the 3 methods.*

Using L'Hopitals Rule: Differentiate both numerator and denominator

$$f(x) = 3 - x^{\frac{1}{2}}, \qquad f'(x) = -\frac{1}{2}x^{-\frac{1}{2}}$$

$$g(x) = 9 - x, \qquad g'(x) = -1$$

$$= \lim_{x\to 9} \frac{-\frac{1}{2}x^{-\frac{1}{2}}}{-1}$$

= =

Exercise: Evaluate the following.

1) $\lim_{x \to 4} \dfrac{x^2 - 2x - 8}{x - 4}$

2)

3) $\lim_{x \to -1} \dfrac{3x^2 - 3}{x + 1}$

Strand	5 **LIMITS CONTINUITY AND DIFFERENTIABILITY**
Sub Strand	5.1.2 Limits of Trigonometric Functions
Content Learning Outcome	Students should be able to; **-** find limit of trigonometric functions.

- For the **Indeterminate Form** in trig functions, you probably have to **use** some Trig Identities to compute limits:
  - $\cos^2 x + \sin^2 x = 1$
  - $\sin 2x = 2 \sin x . \cos x$
  - $\tan x = \dfrac{\sin x}{\cos x}$

- L' Hôpital's rule where applicable.
  Some derivatives are given below:

$y = \sin x$	$y = \cos x$	$y = \cos bx$	$y = \sin bx$
$y' = \cos x$	$y' = -\sin x$	$y' = -b \sin bx$	$y' = b \cos bx$

Example: 1

Evaluate: $\lim_{x \to} \dfrac{sin^2 x + cos^2 x}{x}$

Answer: Using identity $sin^2 x + cos^2 x = 1$

Therefore, $= \lim_{x \to \pi} \dfrac{sin^2 x + cos^2 x}{x}$

Becomes; $= \lim \dfrac{1}{\_}$, directly substituting π,

Example: 2

Evaluate: $\lim_{x \to 0} \dfrac{\sin 2x}{\sin x}$

Directly substitute 0, $= \dfrac{\sin 2(0)}{\sin 0} = \dfrac{0}{0}$

Therefore, replacing $\sin 2x$ gives, becomes

$= \lim_{x \to} \dfrac{2 \sin x \cos x}{\sin}$, $\sin x$ cancels

Example 3: Show that $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Directly substitute $x = 0$, we get $\frac{0}{0}$, so we can use L' Hôpital's Rule

$f(x) = \sin x \rightarrow f'(x) = \cos x$

$g(x) = x \rightarrow g'(x) = 1$

$$= \lim_{x \to 0} \frac{f'(x)}{g'(x)}$$

$$= \lim_{x \to 0} \frac{\cos x}{1}$$

$$= \lim_{x \to 0} \frac{\cos x}{1}$$

$= 1$

Exercise

1. Find the limits of the following identities

$\lim$ $1 - \cos^2 x$ $\lim$ $\cos$ $\lim$ $\sin 2x$

2. **Evaluate the following limits using L' Hôpital's Rule**

a. $\lim_{x \to 0} \frac{2\sin 2x}{5x}$

b. $\lim_{x \to 0} \frac{\cos x - 1}{2x}$

Strand	5 **LIMITS CONTINUITY AND DIFFERENTIABILITY**
Sub Strand	5.1.2 Limits at Infinity
Content Learning Outcome	Students should be able to; - find limit as x approaches ∞.

Notes:

Limits as $x \to \infty$:

- If r is a positive rational number and c is any real number, then

$$\boxed{\lim_{x \to \infty} \frac{c}{x^r} = 0}$$

Division by a very large number gives a very small result.

- To use the above property, divide the numerator and denominator by the **highest power of *x* in the denominator.**

- In other words, <u>use cover up rule.</u>

Example 1: Find $\lim_{x \to \infty} \frac{85}{x^2}$

$\lim_{x \to \infty} \frac{c}{x^r}$ ***, rule indicates when constant is divided by a variable***

$$= \lim_{x \to \infty} \frac{85}{x^2} = 0$$

Example 2: Find $\lim_{x \to \infty} \frac{5x + 2}{x^2 - 4}$

Looking at the expression the variable with the highest power is x^2*. Therefore, we will divides each term of the expression by* x^2*.*

$$= \lim_{x \to \infty} \frac{5x + 2}{x^2 - 4}$$

$$= \lim_{x \to \infty} \frac{\frac{5x}{x^2} + \frac{2}{x^2}}{\frac{x^2}{x^2} - \frac{4}{x^2}}$$

$$= \lim_{x \to \infty} \frac{\frac{5}{x} + \frac{2}{x^2}}{1 - \frac{4}{x^2}}$$ *, now put* ∞ *in place of x.*

$= \lim_{x\to\infty} \dfrac{\frac{5}{\infty}+\frac{2}{\infty}}{1-\frac{4}{\infty}}$, *division by very large number gives a very small result,* $\lim_{x\to\infty}\frac{1}{x}=0$

$= \dfrac{0+0}{1-0} = \dfrac{0}{1} = 0$

Example 3: Find $\lim_{x\to+\infty}\left(3-\dfrac{2}{x+1}\right)$

Answer: $= \lim_{x\to+\infty} 3 - \lim_{x\to+\infty} \dfrac{2}{x+1}$

$= 3 - \lim_{x\to+\infty} \dfrac{\frac{2}{x}}{\frac{x}{x}+\frac{1}{x}}$, *divide each term with the highest power of x*

$= 3 - \lim_{x\to+\infty} \dfrac{\frac{2}{x}}{1+\frac{1}{x}}$

$= 3 - \left(\dfrac{0}{1+0}\right)$

$= 3$

Exercise:

a. *Find* $\lim_{x} \; 5x^3$ b. *Find* $\lim_{x} \; x^5 - 2x^4$

c. *Find* $\lim_{x\to\infty} \frac{x^2+1}{x^3+4}$

d) $\lim_{x\to\infty} \frac{x(4x-3)(7-5x)}{(6x^2-1)(8x-3)}$

Strand	**5.3 Limits, Continuity and Differentiability from Graphs**
Sub Strand	5.3.1 Limits at Infinity

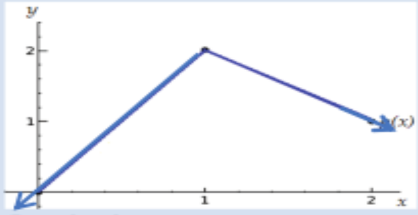
Content Learning Outcome	Students should be able to; **-** find limits, points of discontinuity and points of non – differentiability from graphs.

Limits, Continuity and Differentiability of Piecewise defined Functions

- To find **limit** from a graph, we have to look at the graph at what values it is approaching from the left hand side and right hand side.
  - $\lim_{x \to c^+} f(x)$ (right hand side)
  - $\lim_{x \to c^-} f(x)$ (left hand side)

- **Discontinuity –** A function is discontinuous if it **has a jump** or **has a hole** or it has **asymptote.** Otherwise the function is **continuous.**
- **Non – Differentiable –** A function is non – differentiable if it is **discontinuous**, or **has a sharp corner or has end points**

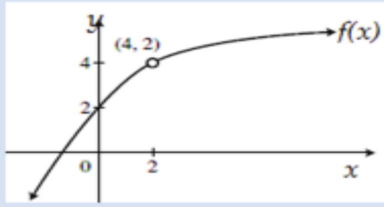
Jump, hole or asymptote



Continuity:
Continuous graph since there is no jump or hole in the graph.

Not differentiable at
$x = 1$(sharp corner)

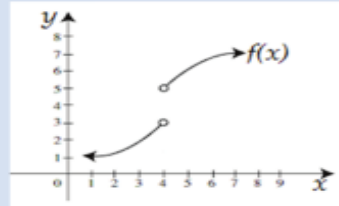
Limits:
Evaluate $\lim_{x \to 1} f(x)$:
Look for the value that y gets close to as x approaches 1 from left and right side
$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 2$
$\therefore \lim_{x \to 1} f(x) = 2$



Continuity:
Discontinuous at $x = 2$, since there is a hole in the graph.

Not differentiable at
$x = 2(discontinuous)$

Limits:
Evaluate $\lim_{x \to 2} f(x)$:
Look for the value that y gets close to as x approaches 2 from left and right side
$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = 4$
$\therefore \lim_{x \to 2} f(x) = 4$

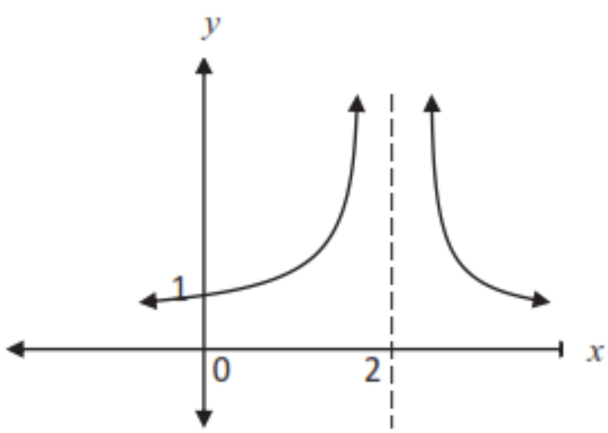


Continuity:
Discontinuous at $x = 4$, since there is a jump.

Not differentiable at
$x = 4(discontinuous)$

Limits:
Evaluate $\lim_{x \to 4} f(x)$:
Look for the value that y gets close to as x approaches 4 from left and right side
$\lim_{x \to 4^-} f(x) = 3 \neq$
$\lim_{x \to 4^+} f(x) = 5$
$\therefore \lim_{x \to 4} f(x)$ *does not exist.*

Example 1 Given below is the graph of $y = f(x)$



Find:

a) $\lim_{x \to 0} f(x)$

To answer this question we have to look at the graph approaching 0 from both left hand side and right hand side. i.e

$\lim_{x \to 0^-} f(x) = 1$ 0^- *means from left hand side. As you can see the arrow in the question approaching from left hand side to the y – value 1. Therefore the answer is 1*

$\lim_{x \to 0^+} f(x) = 1$ 0^+ *means from left hand side. As you can see the arrow in the question approaching from right hand side to the y – value 1. Therefore the answer is 1*

Therefore Answer is $\lim_{x \to 0} f(x) = 1$

b) $\lim_{x \to 2} f(x)$

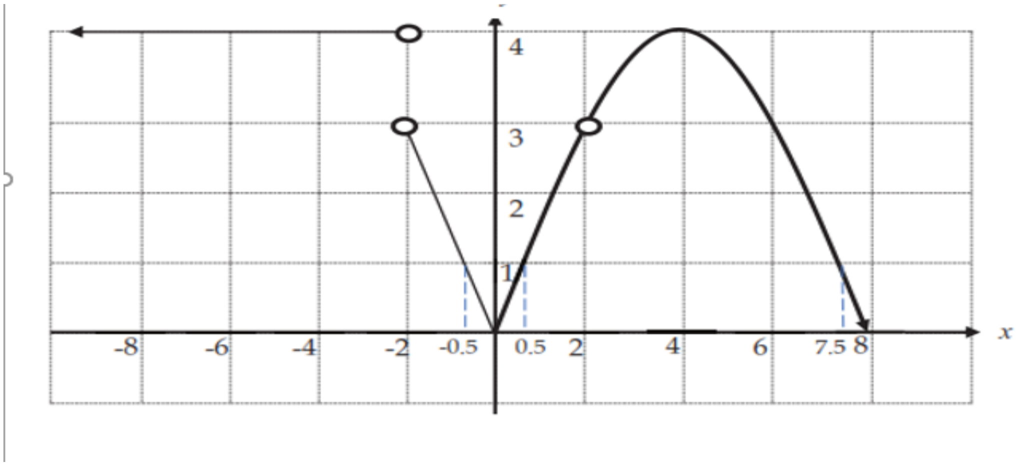
Again to find the limit we have to look at the graph approaching from both the sides

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = +\infty$

y becomes very large as x approaches 2

$\therefore \lim_{x \to 0} f(x) = +\infty$

Example 2: The graph of $g(x)$ **is given below. Use the graph to answer the questions that follow.**



a) Find the following

i) $\lim_{x \to -2^-} g(x) = 4$

ii) $\lim_{x \to -2^+} g(x) = 3$

iii) $\lim_{x \to -2} g(x)$ = limit does not exist

Because it has different values when a approaching from both sides

b) For what values of *x* is

i) $g(x)$ **discontinuous** *(place on the graph where there is jump, hole and asymptote)*

Answer : $x \in \{-2, 2\}$ [graph has hole and jump at the two x - values]

ii) $g(x)$ **not differentiable**. *(place on the graph where there is jump, hole and asymptote and sharp edges)*

Answer: $x \in \{-2, 0, 2\}$ [graph has hole at 2, jump at -2 and sharp edge at 0]

iii) $g(x) = 1$ **?**

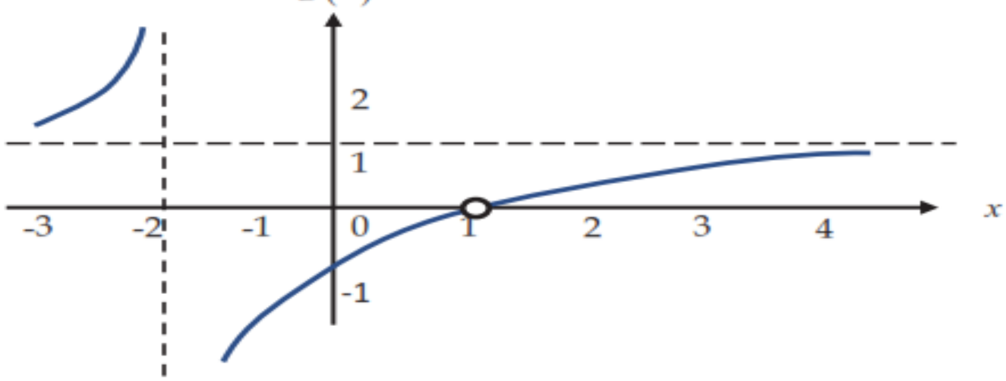
Answer: {-0.5, 0.5, 7.5} [these are the x – values when y=1]

iv) $g(x) = 3$ **?**

Answer: $x = 6$ [when y equals 3]

Exercise

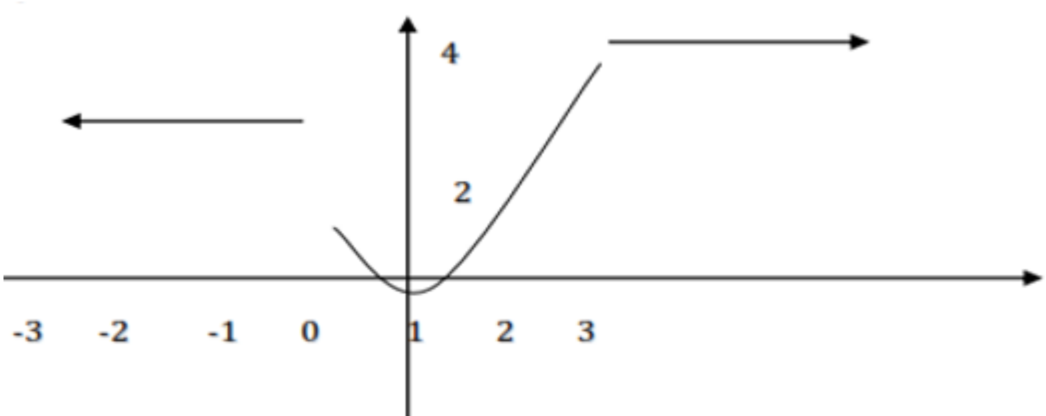
1. The graph of $g(x)$ is given below. Use the graph to answer the questions that follow.



a) Find $\lim_{x \to \infty} g(x)$

b) For what values of x is the function discontinuous?

2.. The graph of piece – wise function h(x) is given below. Use the graph to answer the questions that follow.



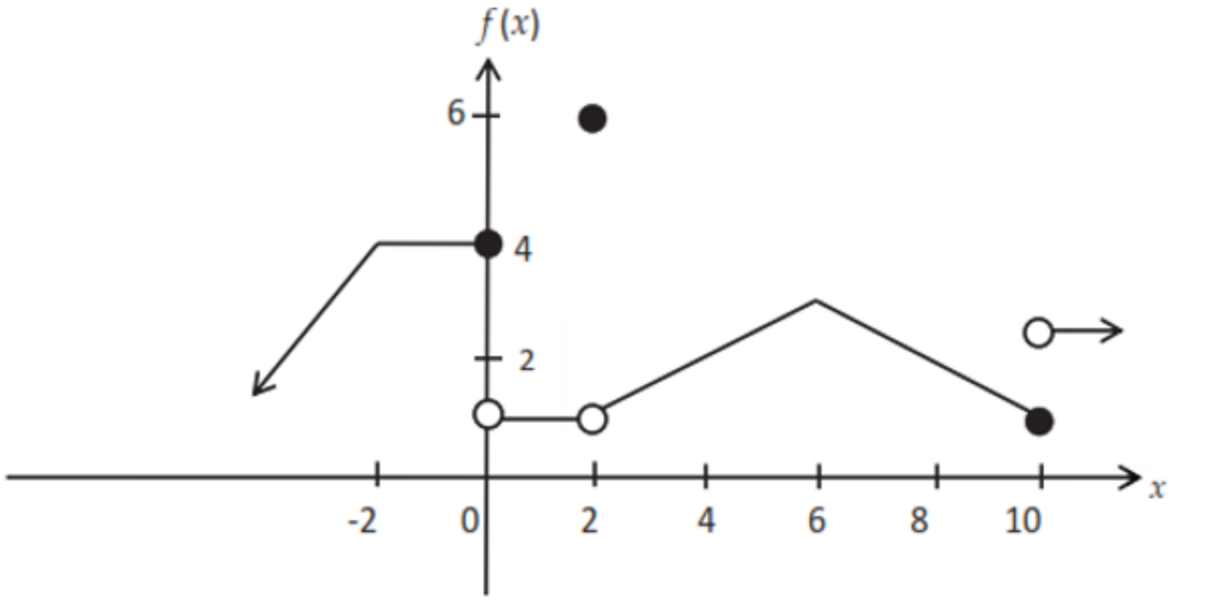
(i) **For what value(s) of x is h(x) discontinuous?**

(ii) **For what value (s) of x is h(x) non-differentiable?**

(iii) **Find** $\lim_{x \to -1} h(x)$

(iv) **Find** $\lim_{x \to 2} h(x)$

3. The graph of function is shown below.



Use the graph to give the value(s) of x for which $f(x)$

a) is continuous but not differentiable.
b) does not have a limit.
c) is equal to 6.

Strand	**6 ALGEBRA**
Sub Strand	6.1.1 Sequences
Content Learning Outcome	- List terms of sequences - List sequence of partial sum - Find limits of a sequence

- **To distinguish between sequences & series:**

(i) ***Sequence*** : numbers separated by commas

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\ldots$

(ii) ***Series*** : obtained by adding numbers in a sequence.

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots$

CONVERGENCE & DIVERGENCE SEQUENCE

- A sequence is converging if *n* approaches infinity.

$$\boxed{\lim_{n \to \infty} A_n = \text{exists}}$$

If the limit exists → *sequence converges*
otherwise the *sequence diverges*

Example 1

A sequence is defined by $T_n = 3n$.

a) Find the first four terms of this sequence.

Answer: *[we can find the terms of the sequence by taking n as 1, 2, 3,4]*

$T_n = 3n$

$T_1 = 3(1) = 3 \quad T_2 = 3(2) = 6 \quad T_3 = 3(3) = 9 \quad T_4 = 3(4) = 12$

The terms of the sequence are <3, 6, 9, 12….>

b) Write as partial sum

Answer: *[to find partial sum, we are going to look at the answer of the terms and add them]*

$S_1 = 3 \quad S_2 = 3 + 6 = 9 \quad S_3 = 3 + 6 + 9 = 18 \quad S_4 = 3 + 6 + 9 + 12 = 30$

The sequence of the partial sum are <3, 9, 18, 30….>

Example 2

A sequence is defined as $a_n = \frac{1}{n^2}$.

a) List the first five terms terms of the sequence.

Answer: *[Take 1, 2, 3, 4, 5 and put it in place of n]*

$$a_1 = \frac{1}{1^2} = 1,\quad a_2 = \frac{1}{2^2} = \frac{1}{4},\quad a_3 = \frac{1}{3^2} = \frac{1}{9},\quad a_4 = \frac{1}{4^2} = \frac{1}{16},\quad a_5 = \frac{1}{5^2} = \frac{1}{25}$$

Therefore the answer is: $< 1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots\ldots >$

b) What is the $\lim_{x \to \infty} \frac{1}{n^2}$?

Answer: $\lim_{x \to \infty} \frac{1}{n^2} = 0$

Exercise:

1. A sequence is defined as $a_n = \frac{2}{n}$.

a) List the first six terms of the sequence.

b) What is the $\lim_{x \to \infty} \frac{2}{n}$?

2. A sequence is defined as $a_n = 3n - 1$.

a. Write the first four terms of the sequence.

b. Write as partial sum.

c. What is the $\lim_{x \to \infty} 3n - 1$?

Strand	**6 ALGEBRA**
Sub Strand	6.1.1 Limits at Infinity
Content Learning Outcome	- List terms of sequences - List sequence of partial sum - Find limits of a sequence - Determine convergence and divergence of sequence.

<u>Convergence and Divergence</u>

Convergence is when sequence approaches a limit, otherwise it is **divergent.**

Example1:

Determine whether the sequence $a_n = \frac{4n + 2}{n - 3}$ **, converge or diverge, and if it converges, then give the value to which it converges to.**

Answer:

[To find the convergence or divergence, we have to find limit as $n \to \infty$*. For this we have to divide all terms with the variable that has the highest power]*

$$\lim_{n\to\infty}\frac{4n+2}{n-3} = \lim_{n\to\infty}\frac{\frac{4n}{n}+\frac{2}{n}}{\frac{n}{n}-\frac{3}{n}}$$

$$= \lim_{n\to\infty}\frac{4+\frac{2}{n}}{1-\frac{3}{n}}$$

$$= \frac{4+\frac{2}{\infty}}{1-\frac{3}{\infty}}$$

$= \dfrac{4+0}{1-0} = 4$ Therefore, it is a **converging** sequence, it **converges to 4.**

Example 2:

Determine whether the sequence $a_n = 2n + 2$**, convergent or divergent?**

[*find limit as* $n\to\infty$.]

Answer: $\lim_{n\to\infty} 2n + 2 = 2\times\infty + 2 = \infty$

Since the terms of the sequence **goes to infinity** (without bounds), therefore it **diverges.**

Exercise:

1. A sequence is defined as $a_n = \dfrac{7n+3}{n-9}$

   a) Find the first four terms of the sequence.

b) Find the first three terms of the partial sum.

c) Find $\lim_{n \to \infty} = \dfrac{7n + 3}{n - 9}$

d) Explain why the sequence converges.

2. A sequence is defined as $a_n = \dfrac{12n + 1}{3n + 2}$

   a. How many terms of the sequence are less than 3.99?

   b. State the limit of the sequence.

   c. Does it converge or diverge? Explain

3. A sequence $a_n = 3n + 4$. Does it converge or diverge? Explain.

Strand	**6 ALGEBRA**
Sub Strand	6.3.1 Binomial Theorem
Content Learning Outcome	- expand using binomial theorem.

- **Factorial !**

The factorial of a number (symbol $n!$) is defined as:

$$\boxed{n! = n \times (n-1) \times (n-2) \times (n-3) \ldots 3 \times 2 \times 1,\ n \in N}$$

Example $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

This can be directly found using the calculator which has the key !

Press 5 ! = ________

- **Combinations** ^nc_r

A combination is a selection of a certain number of elements from a set where the order of elements is not taken into account. The number of possible combinations of "r" elements from "n" things is denoted by:

$$\boxed{^nc_r = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}}$$

Example $^5C_2 = \dfrac{5!}{(5-2)!\,2!} = 10$

This can also be found using the ^nc_r key on the calculator.

Press $5\ ^nc_r\ 2$ = ________

Example 1: Evaluate

a) $\dfrac{8!}{5!}$

$= \dfrac{8 \times 7 \times 6 \times 5!}{5!}$

$= 336$

b)

$= \dfrac{10!}{(10-4)!4!}$

$= \dfrac{10 \times 9 \times 8 \times 7 \times 6!}{4!\ 6!}$

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Binomial Theorem

- The Binomial Theorem

The **binomial theorem** provides a useful method for raising any binomial to a nonnegative integral power:

$$\boxed{(x+a)^n = \binom{n}{0} x^{n-0} a^0 + \binom{n}{1} x^{n-1} a^1 + \binom{n}{2} x^{n-2} a^2 + \ldots\ldots + \binom{n}{n} x^{n-n} a^n}$$

Note:

- The power of x decreases from n to 0 while the power of a increases from 0 to n.
- The number of terms in the expansion is one greater than the power
- The sum of the powers of x and a (first and second term) always equals the power of the binomial n.
- The $(r+1)^{\text{th}}$ term which is the **general term** is given by $T_{r+1} = \binom{n}{r} x^{n-r} a^r$

and $\binom{n}{r}$ is the binomial coefficient.

Example 1: **Expand** $(x + 2)^2$

$= 1 . x^2 . 1 + 2 . x . 2 + 1 . 1 . 4$

$= x^2 + 4x + 4$

Example 2: **Expand** $(2x - 3)^4$

$= 1 \times 16x^4 \times 1 + 4 \times 8 \times x^3 \times -3 + 6 \times 4 \times x^2 \times 9 + 4 \times 2x \times -27 + 1 \times 1 \times 81$

$= 16x^4 - 96x^3 + 216x^2 - 216x + 81$

Example 3: **Expand** $(x^2 - \frac{1}{x})^3$

$$= 1 \cdot x^6 \cdot 1 + 3 \cdot x^4 \cdot \frac{-1}{x} + 3 \cdot x^2 \cdot \frac{1}{x^2} + 1 \cdot 1 \frac{-1}{x^3}$$

$$= x^6 - 3x^3 + 3 - \frac{1}{x^3}$$

Exercise:

1. Use the Binomial Theorem to expand and simplify $(2x + y)^4$.

2. Use the Binomial Theorem to expand and simplify $(2x - \frac{1}{x^2})^3$.

3. A term in an expansion of $(a + b)^c$ is $\binom{n}{k}(3x^2)^4(-2y)^3$.
   a) Give the values of a, b, c, n and k.
   b) Write out the given term in simplified form.

Strand	6 ALGEBRA
Sub Strand	6.3.2 Finding Particular Term
Content Learning Outcome	- find the nth term

Finding the particular term

Example: 1 Find the second term in the expansion of $(x^2 - \frac{1}{y^2})^3$

$$= 3 \times x^4 \times \frac{-1}{y^2}$$

$$= \frac{-3x^4}{y^2}$$

Example: 2 Find the forth term in the expansion of $(2x - 3)^4$

$$= 4 \times 2x \times -27$$

$$= -216x$$

Exercise:

1. Find the third term in the expansion of $(3x - 2)^4$
2. Find the forth term in the expansion of $(x^2 + \frac{1}{x})^6$
3. Find the forth term in the expansion of $(3x^2 - \frac{1}{x})^{15}$

Sub Strand	6.4. Partial Fractions
Content Learning Outcome	- write fractions with distinct linear factors in the denominator as partial fractions.

Type	Factor example	Decomposition
Linear factor	$(x-4)$	$\frac{A}{x-4}$
Repeated linear factor	$(x-4)^2$	$\frac{A}{(x-4)}+\frac{B}{(x-4)^2}$
Quadratic irreducible factor	(x^2+4)	$\frac{Ax+B}{(x^2+4)}$

Before writing a function into a partial fraction ensure that

1. The fraction is bottom heavy
2. The denominator must be in the factored form. This could be

- Linear Factors e.g $(x+2)(x-1)$
- Non factorizable quadratic eg. $(x+2)(x^2+1)$
- Repeated Linear factors eg. $(x+2)(x-1)^2$

6.4.1 Type I - Denominator with distinct linear factors

Follow the following steps while decomposing into partial fraction with distinct linear factors:

1. Factorize the denominator so that you get the distinct linear factors
2. Separate the factors in denominator. Let their constant to be A, B, C, etc
3. Make denominators same.
4. Equate the numerators.
5. Solve for the variables.

Example 1

Express $\dfrac{5x+1}{x^2+x-2}$ as a sum of partial fractions

Factorise x^2+x-2

$$x^2+x-2=(x-1)(x+2)$$

$$\frac{5x+1}{x^2+x-2}=\frac{5x+1}{(x+2)(x-1)}=\frac{A}{x+2}+\frac{B}{x-1}$$

$$\frac{5x+1}{(x+2)(x-1)}=\frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$$

Equate the numerator

$$5x+1=A(x-1)+B(x+2)$$

To eliminate A, let $x=1$

$$5(1)+1=A(1-1)+B(1+2)$$

$$6=3B$$

$$\frac{6}{3}=\frac{3B}{3}$$

$$B=2$$

$$5x+1=A(x-1)+B(x+2)$$

To solve for A, let $x=-2$

$$5(-2)+1=A(-2-1)+B(-2+2)$$

$$-9=-3A$$

$$\boxed{\frac{5x+1}{(x+2)(x-1)}=\frac{3}{x+2}+\frac{2}{x-1}}$$

Example 2:

Express $\dfrac{-2x^2+4x+7}{(x^2-1)(x+3)}$ as a sum of partial fraction.

When factorised: $(x^2-1)=(x-1)(x+1)$

$$\frac{-2(x^2+4x+7)}{(x^2-1)(x+3)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x+3}$$

$$\frac{-2(x^2+4x+7)}{(x^2-1)(x+3)} = \frac{A(x-1)(x+3) + B(x+1)(x+3) + C(x+1)(x-1)}{(x+1)(x-1)(x+3)}$$

Let $x = 1$

$$-2(x^2+4x+7) = A(x-1)(x+3) + B(x+1)(x+3) + C(x+1)(x-1)$$

$$-2((1)^2+4(1)+7) = A(1-1)(1+3) + B(1+1)(1+3) + C(1+1)(1-1)$$

$$-2(1+4+7) = 0 + B(2)(4) + 0$$

$$-24 = 8B$$

$$\frac{-24}{8} = \frac{8B}{8}$$

$$B = -3$$

Let $x = -3$

$$-2(x^2+4x+7) = A(x-1)(x+3) + B(x+1)(x+3) + C(x+1)(x-1)$$

$$-2((-3)^2+4(-3)+7) = A(-3-1)(-3+3) + B(-3+1)(-3+3) + C(-3+1)(-3-1)$$

$$-2(9-12+7) = 0 + 0 + C(-2)(-4)$$

$$-8 = 8C$$

$$\frac{-8}{8} = \frac{8C}{8}$$

$$C = -1$$

Substitute any other value for x to solve for A

Let $x = 0$

$$-2(x^2 + 4x + 7) = A(x-1)(x+3) + B(x+1)(x+3) + C(x+1)(x-1)$$

$$-2((0)^2 + 4(0) + 7) = A(0-1)(0+3) + B(0+1)(0+3) + C(0+1)(0-1)$$

$$-2(7) = A(-1)(3) + B(1)(3) + C(1)(-1)$$

$$-14 = -3A + 3B - C$$

$$-14 = -3A + 3(-3) - (-1)$$

$$-14 = -3A - 9 + 1$$

$$-14 = -3A - 8$$

$$-14 + 8 = -3A - 8 + 8$$

$$-6 = -3A$$

Therefore: $\dfrac{-2(x^2 + 4x + 7)}{(x^2-1)(x+3)} = \dfrac{2}{x+1} - \dfrac{3}{x-1} - \dfrac{1}{x+3}$

$$\frac{-6}{-3} = \frac{-3A}{-3}$$

$$A = 2$$

Exercise:

1. **Express** $\dfrac{2x}{(x-3)(x-5)}$ **as a sum of partial fractions**

2. **Express** $\frac{x - 2}{x(x^2 - 4)}$ **as a sum of partial fractions**

Strand	6 ALGEBRA
Sub Strand	6.4.2 Partial Fractions
Content Learning Outcome	- write fractions with distinct linear factors in the denominator with Repeated Linear Factors

6.4.2 Denominator With Repeated Linear Factors

Repeated Linear Factors – a factor in the denominator that occurs more than once.

Example 1: Express $\frac{x^2}{(x-2)^3}$ **as a sum of partial fractions**

$$(x-2)^3 \to (x-2)(x-2)^2(x-2)^3$$

$$\frac{x^2}{(x-2)^3} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}$$

$$\frac{x^2}{(x-2)^3} = \frac{A(x-2)^2 + B(x-2) + C}{(x-2)^3}$$

$$x^2 = A(x-2)^2 + B(x-2) + C$$

Let $x = 2$;

$$(2)^2 = A(2-2)^2 + B(2-2) + C$$

$$C = 4$$

Let $x = 0$;

Let $x = 1$;

$$(1)^2 = A(1-2)^2 + B(1-2) + C$$

$$1 = A - B + C$$

$$1 = A - B + 4$$

$$A - B = -3$$

$$2(A - B = -3)$$

$$(0)^2 = A(0-2)^2 + B(0-2) + C$$

$$0 = 4A - 2B + C$$

$$0 = 4A - 2B + 4$$

$$4A - 2B = -4$$

$$4A - 2B = -4$$

$$-(2A - 2B = -6)$$

$$2A = 2$$

$$A - B = -3$$

$$1 - B = -3$$

$$-B = -3 - 1$$

Therefore: $$\frac{x^2}{(x-2)^3} = \frac{1}{(x-2)} + \frac{4}{(x-2)^2} + \frac{4}{(x-2)^3}$$

Exercise

1. Express $15 - 4x - x^2$ **as a sum of partial fractions**

2. Express $\dfrac{-4x^2+11x-6}{[illegible]}$ **as a sum of partial fractions**

Strand	6 ALGEBRA
Sub Strand	6.4.3 Partial Fractions
Content Learning Outcome	- write fractions with distinct linear factors in the denominator with quadratics that cannot be factorised

6.4.3 Denominator with Quadratics which cannot be Factorised

- The numerator should have a linear term $Ax + B$
- The degree of the denominator is 2, so the degree of the numerator should 1(linear term)

Example:

Express $\frac{10x + 24}{(x - 3)(x^2 + 9)}$ **as a sum of partial fraction.**

$$\frac{10x + 24}{(x - 3)(x^2 + 9)} = \frac{A}{x - 3} + \frac{Bx + C}{x^2 + 9}$$

$$\frac{10x + 24}{(x - 3)(x^2 + 9)} = \frac{A(x^2 + 9) + (Bx + C)(x - 3)}{(x - 3)(x^2 + 9)}$$

$$10x + 24 = A(x^2 + 9) + (Bx + C)(x - 3)$$

let $x = 0$

$$10x + 24 = A(x^2 + 9) + (Bx + C)(x - 3)$$

$$10(0) + 24 = A((0)^2 + 9) + (B(0) + C)(0 - 3)$$

$$24 = 9A - 3C$$

$A = 3$ $\quad 24 = 9(3) - 3C$

$$24 = 27 - 3C$$

Let $x = 3$;

$$10x + 24 = A(x^2 + 9) + (Bx + C)(x - 3)$$

$$10(3) + 24 = A((3)^2 + 9) + (B(3) + C)(3 - 3)$$

$$54 = 18A$$

$$\frac{54}{18} = \frac{18A}{18}$$

$$A = 3$$

Let $x = 1$:

$$10x + 24 = A(x^2 + 9) + (Bx + C)(x - 3)$$

$$10(1) + 24 = A((1)^2 + 9) + (B(1) + C)(1 - 3)$$

$$34 = 10A - 2B - 2C$$

$$34 = 10(3) - 2B - 2(1)$$

$$34 = 30 - 2B - 2$$

$$\frac{-2B}{-2} = \frac{6}{-2}$$

$$B = -3$$

Therefore:

$$\frac{10x + 24}{(x - 3)(x^2 + 9)} = \frac{3}{x - 3} + \frac{-3x + 1}{x^2 + 9}$$

1. Express $\frac{x}{x(x^2 + 1)}$ as a sum of partial fraction.

2. Express $\dfrac{x-2}{(x+1)(x^2+2)}$ as a sum of partial fraction.