PENANG SANGAM HIGH SCHOOL
P.O.BOX 44, RAKIRAKI
LESSON NOTES – WEEK 22-24

Year/Level: 13 **Subject: Mathematics**

Strand	7 **PROBABILITY AND INFERENTIAL STATISTICS**
Sub Strand	7.4 ESTIMATION
Content Learning Outcome	Students should be able to; - Explain central limit theorem. - Find confidence interval.

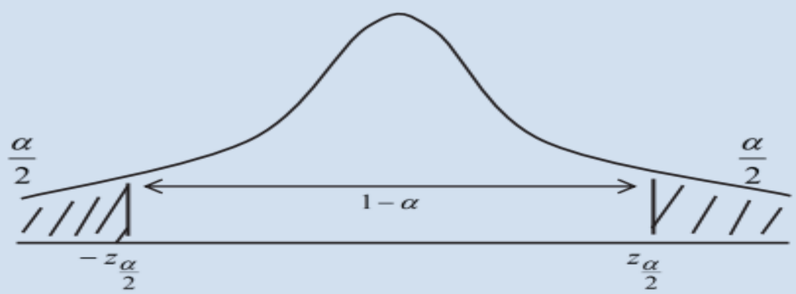
➢ **Central limit theorem**

If a random sample of size n is drawn from a large or infinite population with mean, μ and standard deviation, σ, then the distribution of sample mean, $\bar{x}$, is approximately normally distributed with $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$ and $\mu_{\bar{x}} = \mu$.

Hence $$\boxed{z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}}$$

One very important application of central limit theorem is the determination of the reasonable values of the population mean.
Hypothesis testing and estimation will use the central limit theorem.

Interval Estimate for the population mean



$z_{\frac{\alpha}{2}}$ is the z-value leaving an area of $\frac{\alpha}{2}$ to the right

7.4.2 Confidence Interval

➢ To calculate the Confidence Interval, use the formula

$$\boxed{\bar{x} - z_{\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}}}$$

where:
$\bar{x}$: sample mean
σ : standard deviation
n : sample size
$z_{\sigma/2}$: z-value leaving an area of $\frac{\alpha}{2}$ to the right in a standard normal distribution

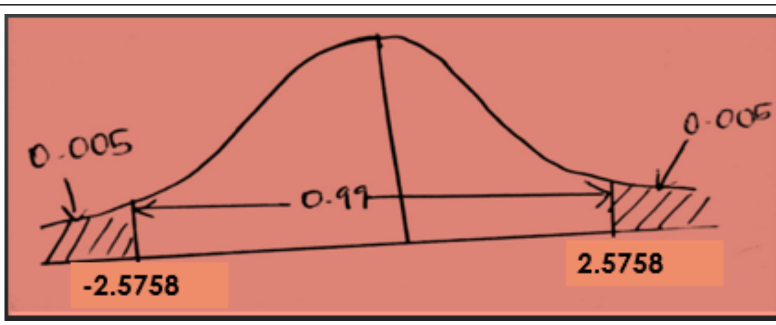
Example 1:

A sample of 100 tourists shows that their mean length of stay is 10 days with a standard deviation of 5 days. Find

95 % confidence interval of μ and state what it means.

This means that there is a probability of 0.95 that the population mean would fall between 9.02 and 10.98.

b) 99% confidence interval



$\alpha = 0.99$

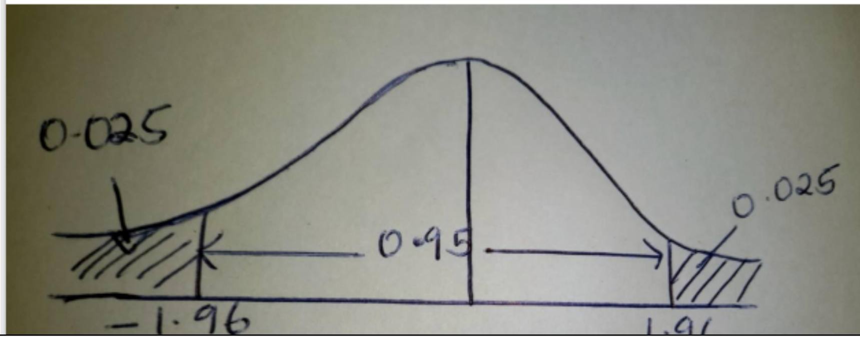
$\frac{\alpha}{2} = \frac{0.99}{2} = 0.495$

$z(p = 0.495) = 2.58$

Hence, using confidence interval formula;

$8.71 < \mu < 11.29$

This means that there is a probability of 0.95 that the population mean would fall between 8.71 and 11.29.



$\alpha = 0.95$

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$

$z(p = 0.475) = 1.96$

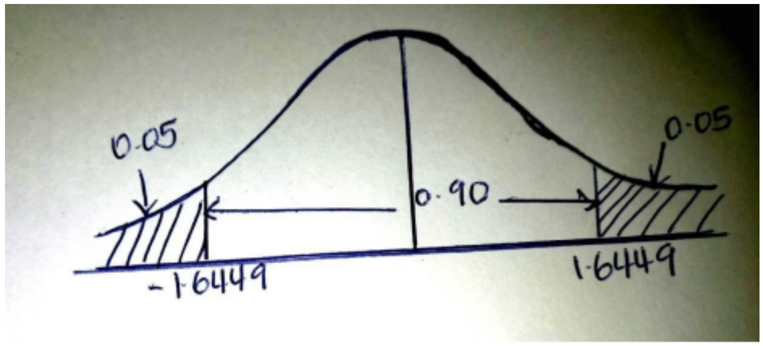
$n = 500$

$\bar{x} = 35$

$\sigma = 12$

$\alpha = 0.90$

$\frac{\alpha}{2} = \frac{0.90}{2} = 0.450$



Hence, using confidence interval formula;

$34.12 < \mu < 35.88$

This means that there is a probability of 0.90 that the population mean would fall between 34.12 and 35.88.

for

Exercise

A sample of 225 students of Penang Sangam High School took a mean time of 30 minutes, with a standard deviation of 6 minutes, to travel to school. Determine the 98% confidence interval for the mean time taken for the students to travel to school.

2. A sample of size 120 items is taken from a population with an unknown mean mass, μ, and standard deviation of 7.7g. The sample mean mass is found to be 562g. Construct a 99% confidence interval for population mean, μ.

Answers:

Strand	7 **PROBABILITY AND INFERENTIAL STATISTICS**
Sub Strand	7.4.3 ESTIMATION – Choosing a Sample Size
Content Learning Outcome	Students should be able to; - Determine sample size for estimating the population mean

7.4.3 Choosing a Sample Size for estimating the population mean

The sample size can be calculated by the formula:

$$n = \left(\frac{z_{\frac{\alpha}{2}}\sigma}{e}\right)^2$$

where

σ = standard deviation

n = sample size

e = error

$z_{\frac{\alpha}{2}}$ = the z - value leaving an area of $\frac{\alpha}{2}$ to the right

The sample size in the formula is the smallest sample size that will satisfy the accuracy requirement. Any larger sample size will also satisfy the requirements.

When finding the sample size, n, all fractional values are rounded up to the next whole number. This will reduce the error.

Examples:

1. Determine the sample size that is required from a population of light bulbs with a bulb life that has a standard deviation of 20 hours, to estimate the mean bulb life to within 5 hours, with 98% confidence.

$n = ?$

$e = 1$

$\sigma = 3$

$\alpha = 0.98$

$\frac{\alpha}{2} = \frac{0.98}{2} = 0.490$

$$n = \left(\frac{z_{\frac{\alpha}{2}} \cdot \sigma}{e}\right)^2$$

$$n = \left(\frac{2.3263\ .\ 20}{5}\right)^2$$

$n = 86.59$

$n = 87$

3. A population has a variance of 16. What sample size should be taken to estimate the mean within two units of the true value within 95% probability?

$n = ?$

$e = 2$

$v = \sigma^2 = 16$ *therfore* $\sigma = 4$

$\alpha = 0.95$

$\frac{\alpha}{2} = \frac{0.95}{2} = 0.475$

$$n = \left(\frac{z_{\frac{\alpha}{2}} \cdot \sigma}{e}\right)^2$$

$$n = \left(\frac{1.96\ .\ 4}{2}\right)^2$$

$n = 15.37$

$n = 16$

4. Determine the sample size that is required from a population with a standard deviation of 5cm to estimate the mean to within 1.4cm with 97% confidence.

$n = ?$

$e = 1.4$

$\sigma = 5$

$\alpha = 0.97$

$\frac{\alpha}{2} = \frac{0.97}{2} = 0.485$

$$n = \left(\frac{z\frac{\alpha}{2} \cdot \sigma}{e}\right)^2$$

$$n = \left(\frac{2.1701 \cdot 5}{1.4}\right)^2$$

$n = 60.07$

$n = 61$

Strand	7 PROBABILITY AND INFERENTIAL STATISTICS
Sub Strand	7.5 Hypothesis Testing
Content Learning Outcome	Students should be able to; - Define terms used in hypothesis testing. - Perform hypothesis testing

7.5.1 Hypothesis Testing

A **hypothesis test** is a statistical test where a sample data is used to decide whether statements made about population parameters are true or false.

Examples of the type of statements to be tested are

- The average price of a school bag in Fiji is \$10.90
- The mean wage of workers in Fiji is \$42 per week.

These statements are examples of **hypothesis**.

An assumption about the **existing situation** or value of a population parameter is called the **null hypothesis** (H_0). This is expressed in the form

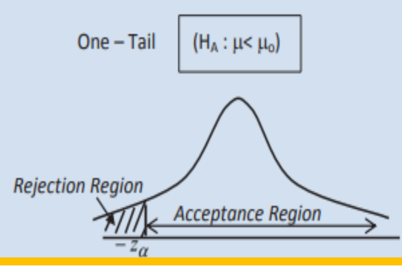
$H_0 : \mu = 42$, if the assumption is that the mean wage of workers is \$42.

The **alternative hypothesis** (H_a) is a new belief about the population parameter which one will have to accept if there is a significant difference between the **sample results** and the **expected results**. This is expressed in either of the forms shown below

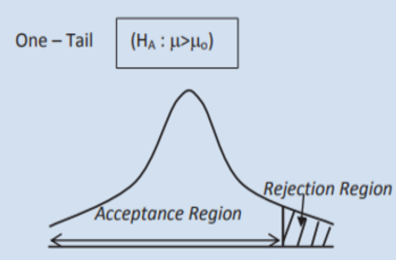
$$H_a : \mu \neq 42 \quad \text{or} \quad H_a : \mu > 42 \quad \text{or} \quad H_a : \mu < 42$$

CASE 1: One – tailed test

- When a '<' sign appears in the alternative hypothesis, the test is called a **left-tailed test**, i.e.

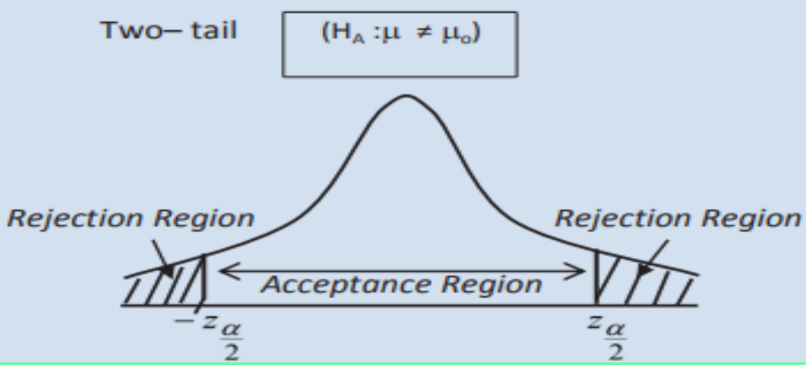


- When a '>' sign appears in the alternative hypothesis, the test is called a **right-tailed test**, i.e.



CASE 2: Two– tailed test

An alternate hypothesis with a ‘ ≠ ’ sign is called a **two-tailed test.**



Steps in Hypothesis Testing

➢ Steps for Hypothesis testing:

- List down all given variables
- Set up the acceptance and rejection regions. Check the alternative hypothesis (H_A) and decide on the type of graph to sketch.
- Use the inverse normal table to find the z-score. (If $\mu > \mu_o$ or $\mu < \mu_o$ find ‘ z_α ’ else if $\mu \neq \mu_o$ find $z_{\frac{\alpha}{2}}$)
- Calculate the value of z:

$$\boxed{z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}}$$

where
$\bar{x}$: sample mean
μ : population mean
σ : standard deviation
n : sample size

- State the conclusion.

Examples:

1. A farmer claims that each bundle of dalo has an average weight of 8kg with a standard deviation of 0.5 kg. A sample of 50 bundles is chosen. It is found that the average weight is 7.8 kg.

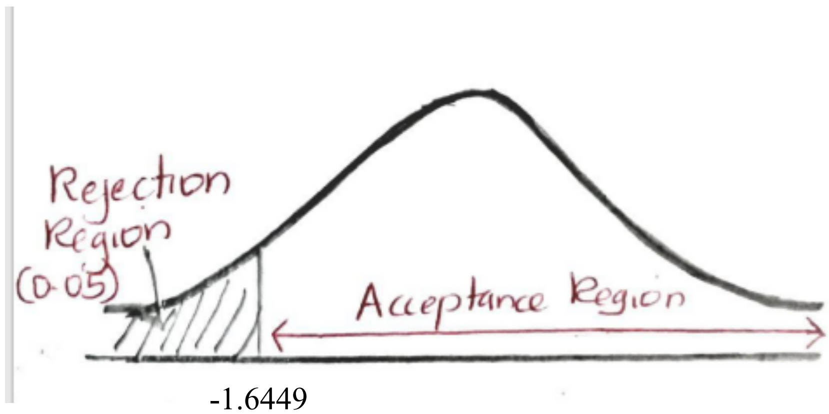
Construct a hypothesis test at 5% level of significance to confirm whether the mean is less than 8 kg (H_o: $\mu = 8$, H_a: $\mu < 8$)

$\bar{x} = 7.8\ kg$

$\sigma = 0.5\ kg$

$n = 50$

$\alpha = 5\,\% = 0.05$



$0.5 - 0.05 = 0.45$

□ = 0.45□= 1.6449

Critical Region: Reject H_o if $z < -1.6449$

$z = -2.83$

Since z value of -2.83 falls on the rejection region we, therefore reject the null hypothesis that H_o: $\mu = 8$

Exercise

2. A researcher claims that Fijian families use an average of 20 coconuts per month with standard deviation of 7. Test the **null hypothesis** H_o: $\mu = 20$ *against the alternative hypothesis*, H_a: $\mu > 20$ if a random sample of 64 Fijian families is found to consume an average of 21 coconuts per month. Use a 0.03 level of significance and state your conclusion clearly.

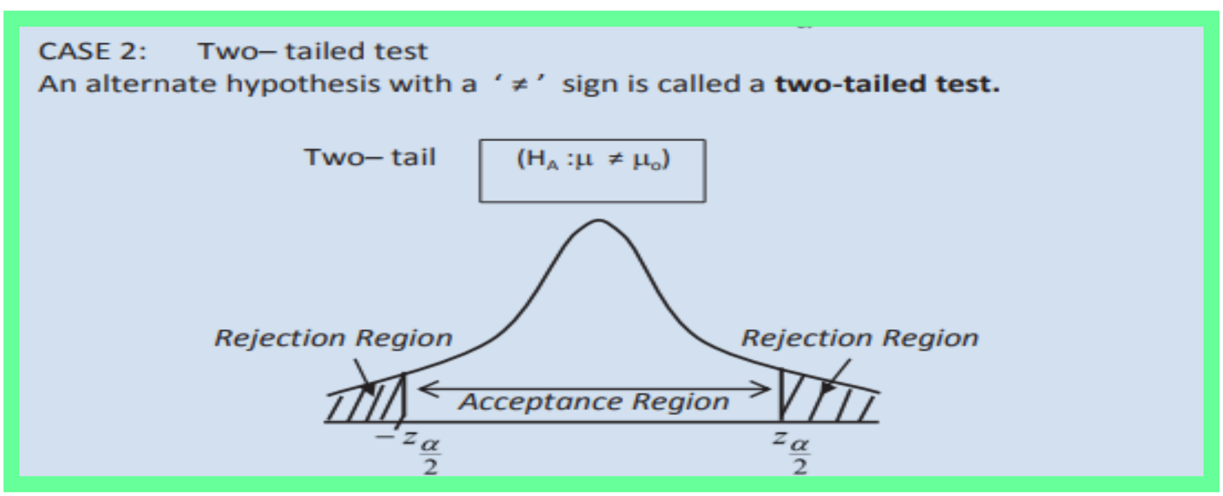
3. The surveys of very large tuna fish caught showed a mean weight of 40kg with a standard deviation of 2.1 kg. Fisherman claimed that because of pollution, the net weight of tuna caught

has decreased. A sample of 80 tuna fish were weighed.

Construct a test in terms of the weight at a 1 % level of significance to determine whether the null hypothesis $H_o\!: \mu = 40\ kg$ can be accepted given the alternative hypothesis $H_o\!: \mu < 40\ kg$.

What conclusion would be reached if the sample mean is 32

Strand	7 **PROBABILITY AND INFERENTIAL STATISTICS**
Sub Strand	7.5 Hypothesis Testing
Content Learning Outcome	Students should be able to; - Define terms used in hypothesis testing. - Perform hypothesis testing using two -tailed test



Example:

A chicken farmer weighed a random sample of 55 chicken birds from his farm. He wanted to test the claim that the mean weight of a chicken at 6 weeks of age in a chicken farm is 1.95 kg with a standard deviation of 0.4 kg.

a. Construct a test at 1% level of significance to determine whether the null hypothesis $H_o: \mu = 1.95\ kg$ can be accepted given the alternative hypothesis $H_A: \mu \neq 1.95\ kg$.

$\bar{x} = 2.3$

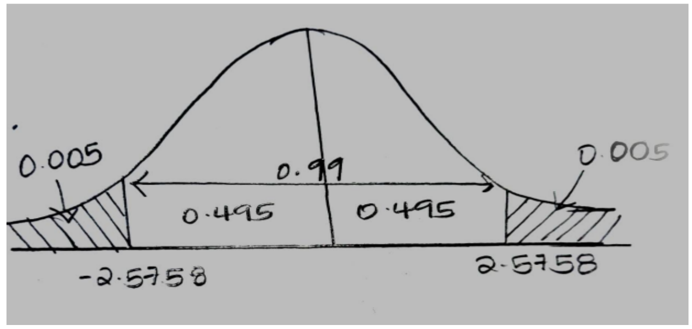
$\sigma = 0.4$

$n = 55$

$\alpha = 0.01 \quad = \frac{\alpha}{2} = \frac{0.01}{2} = 0.005$

$1 - 0.01 = 0.99$

$\frac{0.99}{2} = 0.495$



Critical Region: Reject H_o $if\ z < -2.5758$ and $z > 2.5758$

b. If the sample mean is 2.3 kg, what is your conclusion?

$z = 6.49$

Since z value of 6.49 falls on the rejection region we, therefore reject the null hypothesis that $H_o: \mu = 1.95kg$

Exercise:

1. A company developed a fishing line that it claims has a mean breaking 9kg with a standard deviation of 0.6 kg. A random sample of 50 lines is tested and found to have a mean breaking strength of 8.9 kg.
   Construct a hypothesis test at a 5% level of significance to determine whether the null hypothesis H_o: $\mu = 9\ kg$ can be accepted given the alternative hypothesis H_A: $\mu \neq 9kg$ and state your conclusion.

2. The average hourly wage of a garment factory employee is \$1.05 with a standard deviation of \$0.20. A sample of 50 employees are surveyed and it is found that the average hourly wage is \$0.95. Construct a hypothesis test that H_o: $\mu = \$1.05$ can be accepted given the alternative hypothesis H_A: $\mu \neq \$1.05$ at 3% level of significance and state your conclusion.

Strand	8 DIFFERENTIATIONS

Sub Strand	8.1 Derivatives of Functions
Content Learning Outcome	Students should be able to; - Find derivative of common functions

Rules of Differentiation

- Derivatives of a constant is equal to zero.
- Brackets needs to be expanded.
- Expressions needs to be in the base – index form. eg. $\sqrt{x} = x^{\frac{1}{2}}$
- Variables to be in the numerator e.g $\frac{1}{x^3} = x^{-3}$

$y = x^n$ $y' \text{ or } \frac{dy}{dx} = nx^{n-1}$	$y = ax^n$ $y' \text{ or } \frac{dy}{dx} = anx^{n-1}$

Derivatives of Trigonometric Functions

y	y'
$ln\, x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$-sin^2x$
e^x	e^x

Example:

Differentiate

a. $y = x^3 + 2$

$y' = 3x^2$

b) $f(x) = 5x(x-2) + 4$

$f(x) = 5x^2 - 10x + 4$

$f'(x) = 10x - 10$

c) $f(x) = \frac{5}{x}$

$f(x) = 5x^{-1}$

$f'(x) = -5x^{-1-1}$

$f'(x) = -5x^{-2}$

d)

e) $y = 3\sin 2x$

f) $y = 3\ln x$

$y = x^{\frac{1}{2}}$

$y = \frac{1}{2}x^{\frac{1}{2}-1}$

$y' = \frac{1}{2}x^{-\frac{1}{2}}$

$y' = 3\cos 2x \,.\, 2$

$y' = 6\cos 2x$

$y' = 3 \times \frac{1}{x}$

$y' = \frac{3}{}$

g) $f(x) = -3e^x$

$f'(x) = -3e^x$

Exercise:

Find the derivatives of the following.

a)

b) $g(x) = \frac{1}{3x^3} - 5\cos x$

c) $f(x) = 3x^2 + e^x - 42$

d) $h(x) = \tan x + \ln x$

Strand	8 DIFFERENTIATION
Sub Strand	8.1 Derivatives of Functions
Content Learning Outcome	Students should be able to; - Find derivative of common functions using product rule

Product Rule

Examples

Differentiate:

$y = f\,.\,g$

$y' = f'g + g'f$

1. $y = (3x + 2)(x - 5)$

$f = 3x + 2$

$f' = 3$

$g = x - 5$

$g' = 1$

$y' = f'g + g'f$

$y = 3(x - 5) + 1(3x + 2)$

$= 3x - 15 + 3x + 2$

$= 3xe^x$

$f = 3x$

$f' = 3$

$g = e^x$

$g' = e^x$

$y' = f'g + g'f$

$y' = 3(e^x) + e^x.\ 3x$

3. $f(x) = x^3 \sin x$

$f = x^3$

$f' = 3x^2$

$g = \sin x$

$g' = \cos x$

$y' = f'g + g'f$

$= 3x^2 \sin x + \cos x\,.\ x^3$

$= 3x^2 \sin x + x^3 \cos x$

4.

$f' = \frac{1}{2} x^{-\frac{1}{2}}$

$g = -\ln 2x$

$g' = -\frac{1}{2x}.2 = -\frac{1}{x}$

$y' = f'g + g'f$

Exercise:

Differentiate the following

a. $y = x^2(4x + 1)$

b.

c.

d. $y = \sin x \cos x$

e.

f. $\ln x \sin x$

Strand	**8 DIFFERENTIATION**
Sub Strand	8.1 Derivatives of Functions
Content Learning Outcome	Students should be able to; - Find derivative of common functions using quotient rule

Quotient Rule:

Examples:

$$y = \frac{f}{g}$$

$$y' = \frac{f'g - g'f}{g^2}$$

Find the derivatives o

a. $g(x) = \dfrac{2x-1}{x+3}$

$f = 2x - 1$ $\qquad g = x + 3$

$f' = 2$ $\qquad g' = 1$

$$g'(x) = \frac{2(x+3) - 1(2x-1)}{(x+3)^2}$$

$$= \frac{2x + 6 - 2x + 1}{(x+3)^2}$$

7

$f = 3x^2$ $\qquad g = x + 4$

$f' = 6x$ $\qquad g' = 1$

$$g'(x) = \frac{6x(x+4) - 1(3x^2)}{(x+4)^2}$$

$$= \frac{6x^2 + 24x - 3x^2}{(x+4)^2}$$

$3x^2 + 24x$

c. $y = \dfrac{e^x}{\cos x}$

$f = e^x$ $\qquad g = \cos x$

$f' = e^x$ $\qquad g' = -\sin x$

$$y' = \frac{e^x(\cos x) - -\sin x(e^x)}{(\cos x)^2}$$

$$= \frac{e^x \cos x + e^x \sin x}{(\cos x)^2}$$

Exercise:

Find the derivative of the following

1. $y = \frac{2x+7}{3x-5}$

2.

4. $y = \frac{\sin x}{\cos x}$

5. $y = \frac{e^x}{\ln x}$

THE END